UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 418 Image Processing

# Problem Set 2

Spring 2008

**Issued:** Thursday, January 24, 2008 **Due:** Thursday, January 31, 2007

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## Problem 2.1

Suppose that you need to sample a 10cm×10cm image using at most 10,000 samples, thus, on average, you should have no more than one sample per square millimeter. Other than this constraint, you are free to choose whatever sampling basis $V = [\vec{v}_1, \vec{v}_2]$ gives the least-aliased image.

(a) Prove that the average area per image sample is given by the determinant of $V$. Hint: observe that the sampling area is a parallelogram whose sides are $\vec{v}_1 = [v_{11}, v_{21}]^T$ and $\vec{v}_2 = [v_{12}, v_{22}]^T$.

(b) Your goal is to find a basis $V = [\vec{v}_1, \vec{v}_2]$ such that $|\det(V)| = 1\text{mm}^{-2}$ (one sample per square millimeter). One way to achieve this goal is by constraining the elements of $V$ as follows:

$$v_{11} = r\cos\theta \quad (1)$$
$$v_{21} = r\sin\theta \quad (2)$$
$$v_{12} = r\cos\phi \quad (3)$$
$$v_{22} = r\sin\phi \quad (4)$$

where the units of $r$ are $\text{mm}^{-1}$. Find $r$, as a function of $\theta$ and $\phi$, in order to guarantee that $|\det(V)| = 1\text{mm}^{-2}$.

(c) The continuous-space image, $f_a(x, y)$, is sampled as follows to produce a discrete-space image $f_d[m, n]$:

$$f_d[m, n] = f_a\left(x = mr\cos\theta + nr\cos\phi, y = mr\sin\theta + nr\sin\phi\right) \quad (5)$$

Find $F_d(\omega_1, \omega_2)$ in terms of $F_a(\Omega_1, \Omega_2)$.

(d) Sketch the "Nyquist diamond:" the region, in the $(\Omega_1, \Omega_2)$ plane, in which it is possible for $F_a(\Omega_1, \Omega_2)$ to have energy without being aliased when it is sampled according to Eq. 5. Specify the equations of the four bounding lines, or of their $\Omega_1$ and $\Omega_2$ axis intercepts. Specify these equations in terms of either $\{v_{11}, v_{21}, v_{12}, v_{22}\}$ or $\{r, \theta, \phi\}$.

(e) Suppose that $f_a(x, y)$ is an image with the following Fourier transform:

$$F_a(\Omega_1, \Omega_2) = 2\pi^2\delta(\Omega_1, \Omega_2) + \frac{\pi^2}{2}\left(\delta\left(\Omega_1 - \frac{2\pi}{\lambda}, \Omega_2\right) + \delta\left(\Omega_1 + \frac{2\pi}{\lambda}, \Omega_2\right)\right)$$
$$+\frac{\pi^2}{2}\left(\delta\left(\Omega_1, \Omega_2 - \frac{2\pi}{\lambda}\right) + \delta\left(\Omega_1, \Omega_2 + \frac{2\pi}{\lambda}\right)\right) \quad (6)$$

Find $f_a(x, y)$. Sketch the image, or describe it in words (assume that $f_a(x, y) = 1$ corresponds to a bright spot, and $f_a(x, y) = 0$ corresponds to a dark spot).

(f) Suppose that you have a continuous-space image whose Fourier transform is given in Eq. e, and you wish to sample it using at most one sample per square millimeter. What is the smallest allowable wavelength, $\lambda$, that can be sampled with one sample per square millimeter with no aliasing?